Sequences

A sequence is an ordered list of numbers.
The **sum** of the terms of a sequence is called a **series.**

- Each number of a sequence is called a **term** (or element) of the sequence.
- A **finite sequence** contains a finite number of terms (you can count them). 1, 4, 7, 10, 13
- An **infinite sequence** contains an infinite number of terms (you cannot count them). 1, 4, 7, 10, 13, . . .
- The terms of a sequence are referred to in the subscripted form shown below, where the **natural number subscript** refers to the location (position) of the term in the sequence.

$$\underset{a_1}{1}, \quad \underset{a_2}{4}, \quad \underset{a_3}{7}, \quad \underset{a_4}{10}, \quad \underset{a_5}{13}, \quad \underset{a_6}{16}, \quad \ldots$$

(If you study computer programming languages such as C, C++, and Java, you will find that the first position in their arrays (sequences) start with a subscript of zero.)

- The **general form of a sequence** is represented: $a_1, a_2, a_3, \ldots a_n, \ldots$
- The **domain of a sequence** consists of the counting numbers 1, 2, 3, 4, ... and the **range** consists of the terms of the sequence.
- The terms in a sequence **may, or may not, have a pattern**, or a related formula. For some sequences, the terms are simply random.

Let's examine some sequences that have patterns:
Sequences often possess a **definite pattern** that is used to arrive at the sequence's terms. It is often possible to express such patterns as a formula. In the sequence shown at the left, an **explicit formula** may be:
$a_n = 12n$
where n represents the term's position in the sequence.

$$\underset{12(1)}{\overset{a_1}{12}}, \quad \underset{12(2)}{\overset{a_2}{24}}, \quad \underset{12(3)}{\overset{a_3}{36}}, \quad \ldots$$

Examples:
1. Write the first three terms of the sequence whose n^{th} term is given by the explicit formula:

$$a_n = 2n - 1$$

ANSWER: Remember that n is a natural number (starting with n = 1).

$$a_1 = 2(1) - 1 = 1$$
$$a_2 = 2(2) - 1 = 3$$
$$a_3 = 2(3) - 1 = 5$$

Notice that n is replaced with the number of the term you are trying to find.

2. Find the 5th and 10th terms of the sequence whose n^{th} term is given by:

$$a_n = \frac{n}{n+1}$$

ANSWER: Remember that n corresponds to the location of the term. Use $n = 5$ and $n = 10$.

$$a_5 = \frac{5}{5+1} = \frac{5}{6}$$

$$a_{10} = \frac{10}{10+1} = \frac{10}{11}$$

3. Write an explicit formula for the n^{th} term of a sequence of negative even integers starting with -2.

ANSWER: Get a visual of the terms. -2, -4, -6, -8, ...

Compare the terms to the numbers associated with their locations and look for a pattern.

Notation	Location	Term
a_1	1	-2
a_2	2	-4
a_3	3	-6
a_4	4	-8

Look for a pattern. In this example, each term can be found by multiplying the location number by -2.

A formula could be:

$$a_n = -2n$$

4. Find the first 4 terms of the sequence

$$a_n = (-1)^n(n^2 + 3)$$

$$a_1 = (-1)^1(1^2 + 3) = -4$$
$$a_2 = (-1)^2(2^2 + 3) = +7$$
$$a_3 = (-1)^3(3^2 + 3) = -12$$
$$a_4 = (-1)^4(4^2 + 3) = +19$$

Notice how the terms are alternating signs between negative and positive.

Keep this pattern in mind (involving powers of -1) when asked to write formulas for sequences.

$(-1)^n(n^2 + 3)$ yields -4, 7, -12, 19, ...

$(-1)^{n+1}(n^2 + 3)$ yields 4, -7, 12, -19, ...

Answer the following questions pertaining to sequences.

1.

$a_n = 3n$ term of the sequence

Choose:

3
6
9

2.

$a_n = n^2 - 1$ ms of the sequence:

Choose:

0, 3, 8
3, 8, 15
1, 2, 3

3.

$a_n = n(n+2)$ the sequence

Choose:

146
168
196

4.

Find the 8th term of the sequence

$$a_n = \left(\frac{1}{2}\right)^n$$

Choose:

$\frac{1}{256}$
$\frac{1}{64}$
$\frac{1}{16}$

5.

Write a formula for the sequence
4, 8, 12, 16, 20, ...

Choose:

$a_n = 4n + 1$
$a_n = n + 4$
$a_n = 4n$

6.

Find the 1 $(-1)^{n-1}n^2$ nce

Choose:
196
-225
225

7.

Find the 12th term of the sequence $a_n = (-1)^{n+1}e^{n^2}$

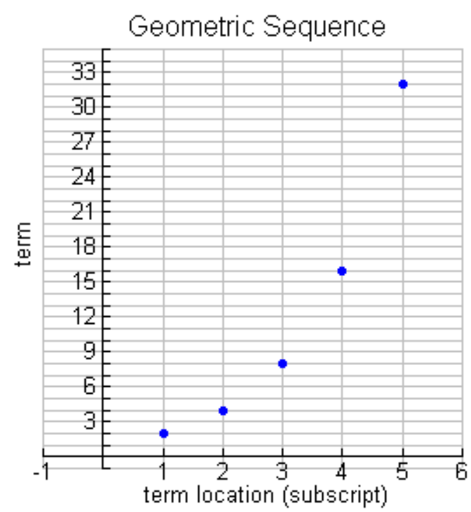
Choose:
-144
144
-e^{144}
e^{144}

Geometric Sequences and Series

A sequence is an ordered list of numbers.
The **sum** of the terms of a sequence is called a **series.**
While some sequences are simply random values,
other sequences have a **definite pattern** that is used to arrive at the sequence's terms.
Two such sequences are the **arithmetic** and **geometric** sequences. Let's investigate the geometric sequence.

Geometric Sequences 

MULTIPLY



If a sequence of values follows a pattern of **multiplyinga fixed amount** (not zero) times each term to arrive at the following term, it is referred to as a **geometric sequence**. The number multiplied each time is constant (always the same).

The fixed amount multiplied is called the **common ratio, *r***, referring to the fact that the ratio (fraction) of the second term to the first term yields this common multiple. To find the common ratio, divide the second term by the first term.

Notice the **non-linear nature** of the scatter plot of the terms of a geometric sequence. The domain consists of the counting numbers 1, 2, 3, 4, ... and the range consists of the terms of the sequence. While the x value increases by a constant value of one, the y value increases by multiples of two (for this graph).

Examples:

Geometric Sequence	Common Ratio, r	
5, 10, 20, 40, ...	$r = 2$	**multiply** each term by 2 to arrive at the next term or...**divide** a_2 by a_1 to find the common ratio, 2.
-11, 22, -44, 88, ...	$r = -2$	**multiply** each term by -2 to arrive at the next term .or...**divide** a_2 by a_1 to find the common ratio, -2.
$4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \frac{64}{81}, \ldots$	$r = \frac{2}{3}$	**multiply** each term by 2/3 to arrive at the next term or...**divide** a_2 by a_1 to find the common ratio, 2/3.

Formulas used with geometric sequences and geometric series:

To **find any term** of a **geometric sequence:**

$$a_n = a_1 \bullet r^{n-1}$$

where a_1 is the first term of the sequence,
r is the common ratio,
n is the number of the term to find.

To find the **sum of a certain number of terms** of a **geometric sequence:**

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where S_n is the sum of n terms (**n^{th} partial sum**),
a_1 is the first term, r is the common ration.

Note: a_1 is often simply referred to as a.

Examples:

Question	Answer
1. Find the common ratio for the sequence $6, \ -3, \ \frac{3}{2}, \ -\frac{3}{4}, \ \ldots$	**1.** The common ratio, r, can be found by dividing the second term by the first term, which in this problem yields **-1/2.** Checking shows that multiplying each entry by -1/2 yields the next entry.
2. Find the common ratio for the sequence given by the formula $a_n = 5(3)^{n-1}$	**2.** The formula indicates that 3 is the common ratio by its position in the formula. A listing of the terms will also show what is happening in the sequence (start with $n = 1$). 5, 15, 45, 135, ... The list also shows the common ratio to be 3.
3. Find the 7th term of the sequence 2, 6, 18, 54, ...	**3.** $n = 7$; $a_1 = 2$, $r = 3$ $a_n = a_1 \bullet r^{n-1}$ $a_7 = 2 \bullet 3^{7-1} = 1458$ The seventh term is 1458.

4. Find the 11^{th} term of the sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$	**4.** $n = 11$; $a_1 = 1, r = -1/2$ $a_{11} = 1 \cdot \left(-\frac{1}{2}\right)^{11-1} = \frac{1}{1024}$
5. Find a_8 for the sequence 0.5, 3.5, 24.5, 171.5, ...	**5.** $n = 8$; $a_1 = 0.5$, $r = 7$ $a_n = a_1 \cdot r^{n-1}$ $a_8 = 0.5 \cdot 7^{8-1} = 411{,}771.5$
6. Evaluate using a formula: $\sum_{k=1}^{5} 3^k$	**6.** Examine the summation $\sum_{k=1}^{5} 3^k = 3^1 + 3^2 + 3^3 + 3^4 + 3^5$ This is a geometric series with a common ratio of 3. $n = 5$; $a_1 = 3, r = 3$ $S_5 = \frac{3(1-3^5)}{1-3} = \frac{-726}{-2} = 363$
7. Find the sum of the first 8 terms of the sequence -5, 15, -45, 135, ...	**7.** The word "sum" indicates a need for the sum formula. $n = 8$; $a_1 = -5, r = -3$ $S_8 = \frac{-5(1-(-3)^8)}{1-(-3)}$ $S_8 = \frac{-5(1-6561)}{4} = \frac{32800}{4} = 8200$
8. The third term of a geometric sequence is 3 and the sixth term is 1/9. Find the first term.	**8.** Think of the sequence as "starting with" 3, until you find the common ratio. $\_\_\_, \_\_\_, \boxed{3, \_\_\_, \_\_\_, \frac{1}{9}}$ *For this modified sequence:* $a_1 = 3$, $a_4 = 1/9$, $n = 4$

	$a_n = a_1 \bullet r^{n-1}$ $\frac{1}{9} = 3 \bullet r^{4-1}$ $\frac{1}{27} = r^3$ $\frac{1}{3} = r$ Now, work backward multiplying by 3 (or dividing by 1/3) to find the actual first term. $a_1 = 27$
9. A ball is dropped from a height of 8 feet. The ball bounces to 80% of its previous height with each bounce. How high (*to the nearest tenth of a foot*) does the ball bounce on the fifth bounce?	**9.** Set up a model drawing for each "bounce". 6.4, 5.12, ___, ___, ___ The common ratio is 0.8. $a_n = a_1 \bullet r^{n-1}$ $a_n = 6.4 \bullet (.8)^{5-1} = 2.62144$ Answer: 2.6 feet